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# Compatible Lie algebroids and the periodic Toda lattice

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**Abstract**

We give the set of maps from $\mathbb{Z}_d$ to $GL(2, \mathbb{R})$ the structure of a Poisson manifold endowed with a pair of compatible Lie algebroids. A suitable reduction process, of the Marsden–Ratiu type, yields a smaller manifold $\mathcal{N}$ with the same geometrical properties as the original manifold. Moreover, $\mathcal{N}$ is a bi-Hamiltonian manifold and the flows naturally defined on it are the periodic Toda flows.


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## 1. Introduction

The (finite) periodic Toda lattice is a system of $d$ particles on a circle, with nearest-neighbor interaction given by an exponential repulsive potential [20]. This system has been studied in depth, both from the physical and from the mathematical point of view. In this paper we investigate the periodic Toda lattice by means of a new approach, which is suggested by the "continuous" case of the KdV equation, i.e., the prototype of the so called *soliton equations* [3]. Indeed, it is well-known that the KdV equation is a bi-Hamiltonian system on an infinite dimensional phase space, and that this phase space is obtained by reduction of the space $\mathrm{Map}(S^1, \mathfrak{gl}(2, \mathbb{R}))$ of $\mathcal{C}^\infty$ maps from $S^1$ to $\mathfrak{gl}(2, \mathbb{R})$. Moreover, the integrability properties of the KdV equation can be easily derived from its bi-Hamiltonian features. To pass from the continuous setting to the discrete one it is quite natural to replace the circle $S^1$ with the cyclic group $\mathbb{Z}_d$. Moreover, it is convenient to replace

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the Lie algebra $\mathfrak{gl}(2,\mathbb{R})$ with its Lie group $GL(2,\mathbb{R})$. This way we are led to consider the finite-dimensional phase space $\mathrm{Map}(\mathbb{Z}_d, GL(2,\mathbb{R}))$. Unfortunately, there is no way to endow this space with a bi-Hamiltonian structure. Therefore in Section 2 we define on $\mathrm{Map}(\mathbb{Z}_d, GL(2,\mathbb{R}))$ a new geometrical structure such that, under some assumptions, a suitable reduction process yields again a bi-Hamiltonian manifold. We call this new kind of manifold a *Poisson bi-anchored manifold*: it is a Poisson manifold equipped with a generalized "Dirac structure" [4] that defines a useful relation on one-forms. More technically, this relation is written in terms of a pair of compatible Lie algebroids [10].

In Section 3 we present the reduction of the specific Poisson bi-anchored manifold $\mathrm{Map}(\mathbb{Z}_d, GL(2,\mathbb{R}))$, which is an adaptation of the Marsden–Ratiu reduction scheme for Poisson manifolds [14]. As we said, the reduced manifold is bi-Hamiltonian and we can apply the theory of Gelfand and Zakharevich [7] to study the integrability properties of the corresponding bi-Hamiltonian flows, as we do in Section 4. These are the flows of the periodic Toda lattice, which appears once more as the discrete counterpart of the KdV equation.

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## 2. Poisson bi-anchored manifolds

In this section we introduce the geometric objects we need for our approach to the Toda lattice. We will endow the manifold $\mathcal{M}$ of the maps from the cyclic group $\mathbb{Z}_d$ to $GL(2,\mathbb{R})$ with several structures, namely a Poisson tensor and two compatible Lie algebroids suitably soldered together.

A point $q$ of $\mathcal{M}$ is simply a $d$-tuple of invertible $2\times 2$ matrices

$$q = (q^1, \dots, q^d), \tag{1}$$

where

$$q^k = \begin{pmatrix} q_1^k & q_2^k \\ q_3^k & q_4^k \end{pmatrix}. \tag{2}$$

We will always be dealing with $d$-tuples of matrices and the following condition is supposed to hold throughout the discussion:

$$(\cdot)^{k+d} = (\cdot)^k. \tag{3}$$

For convenience, we will say that a matrix as in (2) represents a $d$-tuple as in (1). Vector fields on $\mathcal{M}$ will be represented by $d$-tuples of $2\times 2$ matrices $\dot{q}^k$ whose entries are functions of the point $q\in\mathcal{M}$. The same way, one-forms on $\mathcal{M}$ are represented as $d$-tuples of $2\times 2$ matrices $\alpha^k$ where each entry is a function of the point. The value of the one-form $\alpha$ on the vector field $\dot{q}$ is given by the scalar function

$$\langle \alpha, \dot{q} \rangle = \sum_{k=1}^{d} \mathrm{Tr}(\alpha^k \dot{q}^k) = \alpha_1^k \dot{q}_1^k + \alpha_2^k \dot{q}_3^k + \alpha_3^k \dot{q}_2^k + \alpha_4^k \dot{q}_4^k. \tag{4}$$

Next, we endow $\mathcal{M}$ with a Poisson manifold structure, which is inherited from the natural immersion of $GL(2, \mathbb{R})$ in the space of $2 \times 2$ matrices $\mathrm{Mat}(2, \mathbb{R})$ as an open, dense subset. Indeed we can provide $\mathrm{Mat}(2, \mathbb{R})$ with the structure of *twisted Lie algebra* if we define the commutator of two matrices $q_a$ and $q_b$ in $\mathrm{Mat}(2, \mathbb{R})$ as

$$[q_a, q_b] = q_a b q_b - q_b b q_a, \tag{5}$$

where $b$ is any fixed matrix (see, e.g., [16]). Since $\mathrm{Mat}(2, \mathbb{R})$ can be identified with its dual space through the trace form (4), it inherits the canonical (with respect to the commutator (5)) Lie–Poisson bracket on the dual of a Lie algebra defined by Kirillov (see, e.g., [9]). A quick computation shows that the corresponding Poisson tensor $P' : T^*\mathcal{M} \to T\mathcal{M}$ is assigned by the formula

$$\dot{q}^k = P'(\alpha)^k = q^k \alpha^k b - b \alpha^k q^k. \tag{6}$$

In order to recover the Toda lattice we choose

$$b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{7}$$

Then, the Poisson tensor explicitly reads

$$\begin{aligned} \dot{q}_1^k &= q_2^k \alpha_3^k - q_3^k \alpha_2^k, \qquad \dot{q}_2^k = -\alpha_1^k q_2^k - \alpha_2^k q_4^k, \\ \dot{q}_3^k &= q_3^k \alpha_1^k + q_4^k \alpha_3^k, \quad \dot{q}_4^k = 0. \end{aligned} \tag{8}$$

To obtain the Hamiltonian vector field $X_H$ associated with a (Hamiltonian) function $H : \mathcal{M} \to \mathbb{R}$ we simply have to plug its differential $\alpha = \mathrm{d}H$ into Eq. (6).

At this point we go back to the general theory. We define on a Poisson manifold $\mathcal{M}$ an additional structure that represents a generalization of the action of a Lie algebra on $\mathcal{M}$ and was first studied by Mackenzie [10].

**Definition 1.** Let $\mathcal{M}$ be a manifold. Then $(\mathcal{M}, \mathcal{E}, A, \{\cdot,\cdot\})$ is said to be a Lie algebroid if

(a) $\mathcal{E}$ is a vector bundle on $\mathcal{M}$.

(b) $\{\cdot,\cdot\}$ is a bi-linear composition law on $\Gamma(\mathcal{E})$, the space of sections of $\mathcal{E}$, that makes $(\Gamma(\mathcal{E}), \{\cdot,\cdot\})$ into a Lie algebra

$$\{s, t\} + \{t, s\} = 0, \qquad \{\{r, s\}, t\} + \{\{s, t\}, r\} + \{\{t, r\}, s\} = 0. \tag{9}$$

(c) The map $A : \mathcal{E} \to T\mathcal{M}$, called anchor, is a Lie algebra morphism

$$A(\{s, t\}) = [A(s), A(t)], \qquad s, t \in \Gamma(\mathcal{E}), \tag{10}$$

where $[\cdot,\cdot]$ is the usual commutator of vector fields.

If on the same manifold $\mathcal{M}$ and vector bundle $\mathcal{E}$ live two different Lie algebroid structures $(\mathcal{M}, \mathcal{E}, A, \{\cdot,\cdot\})$ and $(\mathcal{M}, \mathcal{E}, A', \{\cdot,\cdot\}')$ we can consider the pencil of composition laws on sections

$$\{s,t\}_\lambda = \{s,t\} + \lambda\{s,t\}', \tag{11}$$

and the pencil of maps

$$A_\lambda(s) = A(s) + \lambda A'(s), \tag{12}$$

where $\lambda$ is a complex parameter.

**Definition 2.** We say that the above Lie algebroids are compatible if $(\mathcal{M}, \mathcal{E}, A_\lambda, \{\cdot,\cdot\}_\lambda)$ is a Lie algebroid for every value of $\lambda$. We call $(\mathcal{M}, \mathcal{E}, A_\lambda, \{\cdot,\cdot\}_\lambda)$ a pencil of Lie algebroids and $A_\lambda$ a pencil of anchors.

It is easy to check that the compatibility condition amounts to

$$[A(s), A'(t)] + [A'(s), A(t)] = A(\{s,t\}') + A'(\{s,t\}). \tag{13}$$

In our approach the two compatible Lie algebroids are important because they allow us to define a useful relation among one-forms.

**Definition 3.** Let $A^*, A'^* : T^*\mathcal{M} \to \mathcal{E}^*$ be the dual anchors of $A$ and $A'$, respectively. We define a one-form $\alpha$ to be related with a one-form $\beta$ (and we denote this by $\alpha \sim \beta$) if

$$A^*\alpha = A'^*\beta.$$

In the example we are considering here, where $\mathcal{M} = \mathrm{Map}(\mathbb{Z}_d, GL(2,\mathbb{R}))$, we can define two compatible Lie algebroids on $\mathcal{M}$ as follows. First, we consider the trivial vector bundle $\mathcal{E} = \mathcal{M} \times [\mathrm{Mat}(2,\mathbb{R})]^d$, where $(\cdot)^d$ denotes the $d$-times Cartesian product $(\cdot) \times \cdots \times (\cdot)$. The sections of this bundle are represented by $d$-tuples of $2 \times 2$ matrices $s^k$ whose entries are functions of the point $q \in \mathcal{M}$. Then we define the pencil of anchors $A_\lambda : \mathcal{E} \to T\mathcal{M}$ as

$$\dot{q}^k = A_\lambda(s)^k = s^{k+1}(q^k + \lambda b) - (q^k + \lambda b)s^k, \tag{14}$$

where $b$ is the constant matrix defined in (7). In components the pencil of anchors reads

$$\begin{aligned}
\dot{q}_1^k &= ((s_1^{k+1} - s_1^k)q_1^k + s_2^{k+1}q_3^k - q_2^k s_3^k) + \lambda(s_1^{k+1} - s_1^k),\\
\dot{q}_2^k &= (s_1^{k+1}q_2^k + s_2^{k+1}q_4^k - s_2^k q_1^k - q_2^k s_4^k) - \lambda(s_2^k),\\
\dot{q}_3^k &= (s_3^{k+1}q_1^k + s_4^{k+1}q_3^k - q_3^k s_1^k - q_4^k s_3^k) + \lambda(s_3^{k+1}),\\
\dot{q}_4^k &= (s_3^{k+1}q_2^k + s_4^{k+1}q_4^k - q_3^k s_2^k - q_4^k s_4^k).
\end{aligned} \tag{15}$$

Let us construct a composition law $\{\cdot,\cdot\}_\lambda$ on $\Gamma(\mathcal{E})$ in such a way that $(\mathcal{M}, \mathcal{E}, A_\lambda, \{\cdot,\cdot\}_\lambda)$ be a pencil of Lie algebroids. Formula (13) and a simple calculation suggest that

$$\{s,t\}_\lambda^k = \partial_{A(s)}t^k - \partial_{A(t)}s^k + [t^k, s^k] + \lambda(\partial_{A'(s)}t^k - \partial_{A'(t)}s^k), \tag{16}$$

where by $\partial_{\dot{q}} t$ we mean the derivative of the section $t$ along the vector field $\dot{q}$. It can be checked that $\{\cdot,\cdot\}_\lambda$ is a Lie bracket on $\Gamma(\mathcal{E})$ for all $\lambda$. Thus, the manifold $\mathcal{M}$ is a pencil of Lie algebroids. To obtain explicitly the relation on the one-forms we need to derive the expression of the dual pencil of anchors $A^*_\lambda = A^* + \lambda A'^*$. An element $\xi$ of the dual vector bundle $\mathcal{E}^*$ can be naturally identified with a point of $\mathcal{E}$ by means of the pairing

$$\langle \xi, s\rangle = \sum_{k=1}^{d} \mathrm{Tr}(\xi^k s^k) = \xi_1^k s_1^k + \xi_2^k s_3^k + \xi_3^k s_2^k + \xi_4^k s_4^k.$$

Therefore the dual pencil $A^*_\lambda : T^*\mathcal{M} \to \mathcal{E}^*$ can be viewed as a map that with a one-form $\alpha$ associates a $d$-tuple $\xi^k$ of matrices whose entries are functions of the point $q \in \mathcal{M}$. A quick calculation shows that $A^*_\lambda$ reads

$$\xi^k = A^*_\lambda(\alpha)^k = (q^{k-1} + \lambda b)\,\alpha^{k-1} - \alpha^k(q^k + \lambda b). \tag{17}$$

With this formula and Definition 3 it is easy to check that $\alpha \sim \beta$ if and only if

$$\begin{aligned}
&q_1^{k-1}\alpha_1^{k-1} + q_2^{k-1}\alpha_3^{k-1} - \alpha_1^k q_1^k - \alpha_2^k q_3^k = \beta_1^{k-1} - \beta_1^k,\\
&q_1^{k-1}\alpha_2^{k-1} + q_2^{k-1}\alpha_4^{k-1} - \alpha_1^k q_2^k - \alpha_2^k q_4^k = \beta_2^{k-1},\\
&q_3^{k-1}\alpha_1^{k-1} + q_4^{k-1}\alpha_3^{k-1} - \alpha_3^k q_1^k - \alpha_4^k q_3^k = -\beta_3^k,\\
&q_3^{k-1}\alpha_2^{k-1} + q_4^{k-1}\alpha_4^{k-1} - \alpha_3^k q_2^k - \alpha_4^k q_4^k = 0.
\end{aligned}$$

Now we return to the general theory to introduce the final character. We recall that so far $\mathcal{M}$ is a Poisson manifold endowed with a pencil of Lie algebroids. We want to solder these structures by means of two maps $J, J' : T^*\mathcal{M} \to \mathcal{E}$ that satisfy the following intertwining conditions: if $\alpha \sim \beta$ then

$$P'(\alpha) = A'(J\alpha + J'\beta), \tag{18}$$

$$P'(\beta) = A(J\alpha + J'\beta). \tag{19}$$

In this case we will say that $\mathcal{M}$ is a Poisson bi-anchored manifold. We resume the whole situation in the following picture:

$$\begin{tikzcd}
\alpha \arrow[rr, "P'"] \arrow[dr, "J"] \arrow[dd, leftrightarrow, "\sim"'] & & X \\
& J(\alpha) + J'(\beta) \arrow[ur, "A'"] \arrow[dr, "A"] & \\
\beta \arrow[rr, "P'"] \arrow[ur, "J'"] & & Y
\end{tikzcd}$$

A class of Poisson bi-anchored manifolds are for example the Poisson–Nijenhuis (PN) manifolds (see [13]). These are manifolds $\mathcal{M}$ endowed with a zero-torsion tensor $N : T\mathcal{M} \mapsto T\mathcal{M}$ and a Poisson structure $P$, together with suitable compatibility conditions.

In that case the vector bundle $\mathcal{E}$ on which the anchors are defined is the tangent bundle, $T\mathcal{M}$; the two anchors are the Nijenhuis tensor $N$ and the identity map respectively; the composition laws on the vector fields are easily obtained in terms of the Nijenhuis tensor; the intertwining conditions are also satisfied, as a consequence of the compatibility conditions between the Nijenhuis and the Poisson structure of a PN-manifold.

In the case of PN-manifolds the relation (3) among two one-forms is summarized by the dual tensor $N^* : \alpha \sim \beta$ if $\beta = N^*\alpha$. This relation displays a very nice property: if the image through $N^*$ of an exact one-form $\mathrm{d}f \in \mathfrak{X}^*(M)$ is exact, i.e., there exists a function $f_1$ such that

$$N^*\mathrm{d}f = \mathrm{d}f_1, \tag{20}$$

then the recursive application of the operator $N^*$ to $\mathrm{d}f$ generates a bi-Hamiltonian hierarchy. This is why the PN-manifolds represent a natural setting to study integrable Hamiltonian systems.

Going back to our example, if we define the maps $J, J' : \alpha \to s$ by

$$J : s^k = \alpha^k q^k, \qquad J' : s^k = -\alpha^k b,$$

it is easy to see that they verify the intertwining conditions (18) and (19). In components the maps read

$$J : \begin{cases} s_1^k = \alpha_1^k q_1^k + \alpha_2^k q_3^k, \\ s_2^k = \alpha_1^k q_2^k + \alpha_2^k q_4^k, \\ s_3^k = \alpha_3^k q_1^k + \alpha_4^k q_3^k, \\ s_4^k = \alpha_3^k q_2^k + \alpha_4^k q_4^k, \end{cases} \qquad J' : \begin{cases} s_1^k = -\beta_1^k, \\ s_2^k = 0, \\ s_3^k = -\beta_3^k, \\ ss_4^k = 0. \end{cases}$$

This ends the definition of the geometrical structures of the manifold $\mathcal{M} = \mathrm{Map}(\mathbb{Z}_d, GL(2, \mathbb{R}))$: it is a Poisson manifold endowed with two compatible Lie algebroid structures that define a relation on one-forms and two intertwining maps that solder everything.

## 3. The reduction

In this section we will leave the general theory. We will operate a reduction of the manifold $\mathcal{M} = \mathrm{Map}(\mathbb{Z}_d, GL(2, \mathbb{R}))$ studied in the previous section. With a combination of a restriction and a projection we will obtain a new manifold $\mathcal{N}$ of lower dimension with the same geometrical structure as the original manifold $\mathcal{M}$. The interesting point is that on $\mathcal{N}$ the Poisson tensor combines with the relation on one-forms in such a way to define a second Poisson tensor. A check of the compatibility between the two Poisson tensors will make $\mathcal{N}$ into a bi-Hamiltonian manifold, which is the object of interest in our approach (see, e.g., [11]). The reduction process we present here follows the reduction scheme (for Poisson manifolds) of Marsden and Ratiu [14].

As a first step, we consider the distribution $\mathrm{Im}\,P' + \mathrm{Im}\,A'$. It is easy to verify by formulas (6) and (14) that this distribution is integrable and therefore it foliates $\mathrm{Map}(\mathbb{Z}_d, GL(2,\mathbb{R}))$ in maximal integral leaves, which are the $3d$-dimensional hyperplanes of the form

$$q^k = \begin{pmatrix} q_1^k & q_2^k \\ q_3^k & \nu^k \end{pmatrix}, \tag{21}$$

where $\nu^k$ is a constant. The restriction process we mentioned above consists in selecting one of these leaves. To obtain the Toda lattice, we pick the leaf $\mathcal{L}$ defined by $\nu^k = 0$ for all $k$.

The pencil of anchors allows us to define another distribution, namely $D = A(\ker A')$, which is integrable. [1] We are interested only in the restriction of this distribution to the leaf $\mathcal{L}$, which we denote by $D|_{\mathcal{L}}$. The distribution $E = D|_{\mathcal{L}} \cap T\mathcal{L}$ of $\mathcal{L}$ is also integrable and an explicit computation shows that $E$ is spanned by the vector fields of the form

$$\dot q_1^k = 0, \qquad \dot q_2^k = -\mu^k q_2^k, \qquad \dot q_3^k = \mu^{k+1} q_3^k, \tag{22}$$

for arbitrary $\mu^k$. From this expression we see that along the vector fields in $E$ the following equations are satisfied:

$$\dot q_1^k = 0, \qquad (q_2^{k+1} q_3^k)^{\bullet} = 0. \tag{23}$$

This means that the distribution $E$ admits the two invariants

$$a_1^k = q_1^k, \quad a_2^k = q_2^{k+1} q_3^k. \tag{24}$$

At this point we can operate the projection we mentioned above: we define the reduced manifold $\mathcal{N}$ to be the quotient of the leaf $\mathcal{L}$ with respect to the foliation induced by the distribution $E$. By (23) and (24) we see that $\mathcal{N}$ is a $2d$-dimensional manifold that can be regarded as $\mathbb{R}^{2d}$, endowed with the set of coordinates $(a_1^k, a_2^k)_{k=1,\dots,d}$. The above formulas also yield the expression of the canonical projection $\pi : \mathcal{L} \to \mathcal{N}$.

After obtaining the reduced manifold $\mathcal{N}$ we endow it with a Poisson structure. This can be done observing that $(\mathcal{M}, P', D, \mathcal{L})$ is Poisson reducible, in the terminology of [14]. To find the expression of the reduction of $P'$ we have to extend a generic one-form $\varphi$ on $\mathcal{N}$ to a one-form $\alpha$ on $\mathcal{M}$ (possibly defined only at the points of the leaf $\mathcal{L}$) which annihilates the distribution $D$. This means

$$\langle \alpha, D\rangle = 0, \qquad \langle \alpha, \dot q\rangle = \langle \varphi, \pi_* \dot q\rangle.$$

Let us denote by $\alpha = \mathrm{ext}(\varphi)$ any such extension. Then the expression

$$\mathfrak{p}'(\varphi) = \pi_* \circ P'(\mathrm{ext}(\varphi)) \tag{25}$$

does not depend on the choice of $\mathrm{ext}(\varphi)$ and determines a Poisson structure on $\mathcal{N}$. If we denote by $\varphi = \sum_{k=1}^d \varphi_1^k \mathrm{d}a_1^k + \varphi_2^k \mathrm{d}a_2^k$ the generic one-form on $\mathcal{N}$, an easy calculation shows

[1] This is true in general, provided that for any two sections $s, t$ in $\Gamma(\mathcal{E})$ such that $A'(s) = 0$ and $A'(t) = 0$ we have $\{s,t\}' = 0$. It is evident from (16) that this condition holds in our case.

that an extension $\alpha = \mathrm{ext}(\varphi)$ has the form

$$\alpha^k = \begin{pmatrix} \varphi_1^k & q_2^{k+1}\varphi_2^k \\ q_3^{k-1}\varphi_2^{k-1} & \alpha_4^k \end{pmatrix}. \tag{26}$$

As we said, this matrix is not completely determined: the components $\alpha_4^k$ are free, since the extension gives an equivalence class of one-forms. Now we can apply formula (25) to obtain the expression of the reduced Poisson tensor $\mathfrak{p}'$

$$\dot{a}_1^k = a_2^{k-1}\varphi_2^{k-1} - a_2^k\varphi_2^k, \qquad \dot{a}_2^k = a_2^k(\varphi_1^k - \varphi_1^{k+1}).$$

This is the well-known first Poisson structure of the periodic Toda lattice [1].

To find another Poisson structure we have to determine the reduced relation on the one-forms of $\mathcal{N}$. Therefore we need the expression of the reduced dual pencil of anchors $\mathfrak{a}_\lambda^*$. To have this, in turn, we have to define a proper reduced vector bundle $\mathcal{U}$ based on $\mathcal{N}$ on which the reduced anchors act: $\mathfrak{a}_\lambda : \mathcal{U} \to T\mathcal{N}$. It is convenient to define first the dual vector bundle $\mathcal{U}^*$ and the dual pencil $\mathfrak{a}_\lambda^*$. The definition of the vector bundle $\mathcal{U}$ and the pencil $\mathfrak{a}_\lambda$ will then follow by duality.

Let us evaluate the original dual pencil of anchors $A_\lambda^*$ on the general extension $\alpha$ of the one form $\varphi \in \mathfrak{X}^*(\mathcal{N})$. An explicit calculation shows that $\xi = A_\lambda^*(\mathrm{ext}(\varphi))$ is given in components by

$$\begin{aligned} \xi_1^k &= \varphi_1^{k-1}(q_1^{k-1}+\lambda) - \varphi_1^k(q_1^k+\lambda) + \varphi_2^{k-2}q_2^{k-1}q_3^{k-2} - \varphi_2^k q_2^{k+1}q_3^k, \\ \xi_2^k &= -\varphi_1^k q_2^k + \varphi_2^{k-1}q_2^k(q_1^{k-1}+\lambda) + \alpha_4^{k-1}q_2^{k-1}, \\ \xi_3^k &= \varphi_1^{k-1}q_3^{k-1} - \varphi_2^{k-1}q_3^{k-1}(q_1^k+\lambda) - \alpha_4^k q_3^k, \qquad \xi_4^k = 0. \end{aligned} \tag{27}$$

We notice that the following combinations:

$$\begin{aligned} \xi_1^k &= \varphi_1^{k-1}(q_1^{k-1}+\lambda) - \varphi_1^k(q_1^k+\lambda) + \varphi_2^{k-2}q_2^{k-1}q_3^{k-2} - \varphi_2^k q_2^{k+1}q_3^k, \\ \xi_2^{k+1}q_3^k + \xi_3^k q_2^k &= \varphi_1^{k-1}q_2^k q_3^{k-1} - \varphi_1^{k+1}q_2^{k+1}q_3^k + q_2^{k+1}q_3^k\varphi_2^k(q_1^k+\lambda) \\ &\quad - q_2^k q_3^{k-1}\varphi_2^{k-1}(q_1^k+\lambda), \end{aligned}$$

do not depend on the choice of the extension, i.e., they do not depend on $\alpha_4^k$, and are invariant along the fibers of the projection, i.e., they depend on the coordinates of $\mathcal{L}$ only via the combinations $q_1^k = a_1^k$ and $q_2^{k+1}q_3^k = a_2^k$. Therefore we define the reduced dual vector bundle as $\mathcal{U}^* = \mathcal{N} \times \mathbb{R}^{2d}$, with coordinates $(a_1^k, a_2^k, \eta_1^k, \eta_2^k)$, and we assign the projection $\tau : \mathcal{E}^*|_{\mathcal{L}} \to \mathcal{U}^*$ as follows: on the basis it is the canonical projection $\pi : \mathcal{L} \to \mathcal{N}$ defined by (23) and (24); on the fibers it reads

$$\eta_1^k = \xi_1^k, \tag{28}$$

$$\eta_2^k = \xi_2^{k+1}q_3^k + \xi_3^k q_2^k. \tag{29}$$

Hence the reduced dual pencil of anchors, which is defined in analogy with (25) as

$$\eta = \mathfrak{a}_\lambda^*(\varphi) = (\tau \circ A_\lambda^*)(\mathrm{ext}(\varphi)), \tag{30}$$

reads explicitly

$$\begin{aligned}\eta_1^k &= \varphi_1^{k-1}(a_1^{k-1}+\lambda) - \varphi_1^k(a_1^k+\lambda) + \varphi_2^{k-2}a_2^{k-2} - \varphi_2^k a_2^k,\\ \eta_2^k &= \varphi_1^{k-1}a_2^{k-1} - \varphi_1^{k+1}a_2^k + a_2^k\varphi_2^k(a_1^k+\lambda) - a_2^{k-1}\varphi_2^{k-1}(a_1^k+\lambda).\end{aligned} \tag{31}$$

Expression (31) also yields the relation on the one-forms of $\mathcal{N}$. By Definition 3 we have that two one-forms $\varphi$, $\psi$ on $\mathcal{N}$ are related if

$$\mathfrak{a}^*\varphi = \mathfrak{a}'^*\psi, \tag{32}$$

which in components reads

$$\begin{aligned}&\varphi_1^k a_1^k - \varphi_1^{k+1}a_1^{k+1} + \varphi_2^{k-1}a_2^{k-1} - \varphi_2^{k+1}a_2^{k+1} = \psi_1^k - \psi_1^{k+1},\\ &a_1^k(a_2^k\varphi_2^k - a_2^{k-1}\varphi_2^{k-1}) + \varphi_1^{k-1}a_2^{k-1} - \varphi_1^{k+1}a_2^k = a_2^k\psi_2^k - a_2^{k-1}\psi_2^{k-1}.\end{aligned}$$

To complete the Poisson bi-anchored structure of the reduced manifold $\mathcal{N}$ we have to calculate the reduced pencil of anchors $\mathfrak{a}_\lambda : \mathcal{U} \to T\mathcal{N}$, which is readily obtained by duality, and the reduced intertwining maps $\mathfrak{j}, \mathfrak{j}'$. This computation is a bit more complicated and can be found in [15]. If we choose the set of coordinates $(a_1^k, a_2^k, u_1^k, u_2^k)$ for the vector bundle $\mathcal{U}$, then the intertwining maps turn out to be

$$\mathfrak{j}: \begin{cases} u_1^k = -\varphi_2^{k-1}a_2^{k-1}, \\ u_2^k = \varphi_1^k, \end{cases} \qquad \mathfrak{j}': \begin{cases} u_1^k = 0, \\ u_2^k = 0. \end{cases}$$

Now we focus on the special feature of the *reduced* Poisson bi-anchored manifold $\mathcal{N}$. By Eq. (32), for a fixed one-form $\varphi$ there is a whole class of one-forms $[\psi] = \psi + \ker\mathfrak{a}'^*$ that is related with it. In the reduced structure, though, $\ker\mathfrak{a}'^* \subset \ker\mathfrak{p}'$. Therefore the following tensor

$$\mathfrak{p}(\varphi) := \mathfrak{p}'(\psi) \tag{33}$$

is well defined. A lengthy calculation shows that the bracket induced by this tensor verifies the Jacobi identity. Furthermore, $\mathfrak{p}$ is compatible with $\mathfrak{p}'$, i.e., the pencil $\mathfrak{p}_\lambda = \mathfrak{p} + \lambda\mathfrak{p}'$ is a Poisson tensor [2] for all values of the complex parameter $\lambda$. Explicitly $\dot{a} = \mathfrak{p}_\lambda(\varphi)$ reads

$$\begin{aligned}\dot{a}_1^k &= (\varphi_2^{k-1}a_2^{k-1} - \varphi_2^k a_2^k)(a_1^k+\lambda) + \varphi_1^{k+1}a_2^k - \varphi_1^{k-1}a_2^{k-1},\\ \dot{a}_2^k &= a_2^k(\varphi_1^k(a_1^k+\lambda) - \varphi_1^{k+1}(a_1^{k+1}+\lambda) + \varphi_2^{k-1}a_2^{k-1} - \varphi_2^{k+1}a_2^{k+1}).\end{aligned} \tag{34}$$

Hence we recovered the bi-Hamiltonian structure of the periodic Toda lattice (see [17] and references therein). Our goal has been achieved: we arrived at a bi-Hamiltonian manifold by means of a systematic procedure of reduction of the original Poisson bi-anchored manifold. Now we have to investigate the information provided by the bi-Hamiltonian structure.

[2] We recall that a manifold endowed with two compatible Poisson tensors is said to be bi-Hamiltonian.

## 4. The periodic Toda lattice

In this section we show how the bi-Hamiltonian structure obtained in the previous section accounts for the integrability of the periodic Toda system. In general, to obtain an integrable system from a bi-Hamiltonian manifold $(\mathcal{M}, P, P')$ it is convenient to focus on the Casimirs of the Poisson pencil, i.e., the functions $C$ such that their differentials are in the kernel of $P_\lambda$. Indeed, if we expand a Casimir in powers of $\lambda$

$$C = \sum_i H_i \lambda^i, \tag{35}$$

it is immediate to check that the coefficients $H_i$ satisfy the Lenard relations

$$P'(\mathrm{d}H_i) = -P(\mathrm{d}H_{i+1}). \tag{36}$$

It is easily shown (see, e.g., [11]) that this in turn implies that the $H_i$'s are in involution with respect to either Poisson bracket. If these coefficients are enough, the system is integrable in the classical sense of Arnold [2]. Nevertheless, finding the Casimirs of a Poisson pencil is not easy in general. In the present case we can make use of the following proposition

**Proposition 4.** *If $h_k$ solves*

$$h_k h_{k+1} = (a_1^{k+1} + \lambda) h_k + a_2^k, \tag{37}$$

*then*

$$C(\lambda) = h_1 \cdots h_d$$

*is a Casimir of the Poisson pencil* (34). *The solutions $h_k$ and thus the Casimirs $C$ can be calculated explicitly as Laurent series in the parameter $\lambda$.*

**Proof.** We divide the proof in three parts.

1. By (34) a one-form $\varphi$ in the kernel of $\mathfrak{p}_\lambda$ must satisfy

$$\varphi_1^k(a_1^k + \lambda) - \varphi_1^{k+1}(a_1^{k+1} + \lambda) + \varphi_2^{k-1}a_2^{k-1} - \varphi_2^{k+1}a_2^{k+1} = 0, \tag{38}$$

$$(\varphi_2^{k-1}a_2^{k-1} - \varphi_2^k a_2^k)(a_1^k + \lambda) + \varphi_1^{k+1}a_2^k - \varphi_1^{k-1}a_2^{k-1} = 0. \tag{39}$$

Eq. (38) implies

$$\varphi_1^k(a_1^k + \lambda) + \varphi_2^{k-1}a_2^{k-1} + \varphi_2^k a_2^k = L, \tag{40}$$

where $L$ is an arbitrary function that does not depend on the site $k$. On the other hand, Eq. (39) implies

$$\begin{aligned} 0 &= (\varphi_2^{k-1}a_2^{k-1} - \varphi_2^k a_2^k)\varphi_1^k(a_1^k + \lambda) + \varphi_1^k\varphi_1^{k+1}a_2^k - \varphi_1^k\varphi_1^{k-1}a_2^{k-1} \\ &= (\varphi_2^{k-1}a_2^{k-1} - \varphi_2^k a_2^k)(L - \varphi_2^{k-1}a_2^{k-1} - \varphi_2^k a_2^k) + \varphi_1^k\varphi_1^{k+1}a_2^k - \varphi_1^k\varphi_1^{k-1}a_2^{k-1} \\ &= L(\varphi_2^{k-1}a_2^{k-1} - \varphi_2^k a_2^k) - (\varphi_2^{k-1}a_2^{k-1})^2 + (\varphi_2^k a_2^k)^2 - \varphi_1^k\varphi_1^{k-1}a_2^{k-1} + \varphi_1^k\varphi_1^{k+1}a_2^k, \end{aligned}$$

and thus it is equivalent to

$$(\varphi_2^k a_2^k)^2 + \varphi_1^k \varphi_1^{k+1} a_2^k - L\varphi_2^k a_2^k = G, \tag{41}$$

where $G$ is another arbitrary function that does not depend on the site $k$. In order to select only *exact* one-forms we set $L$ to be a constant and $G = 0$. Under these assumptions, the system (38) and (39) is equivalent to

$$\varphi_1^k(a_1^k + \lambda) + \varphi_2^{k-1} a_2^{k-1} + \varphi_2^k a_2^k = L, \tag{42}$$

$$a_2^k(\varphi_2^k)^2 + \varphi_1^k \varphi_1^{k+1} - L\varphi_2^k = 0. \tag{43}$$

If we set

$$h_k = \frac{\varphi_1^{k+1}}{\varphi_2^k}, \tag{44}$$

we can rewrite the system (38) and (39) as

$$h_{k-1}(a_1^k + \lambda) + a_2^{k-1} = \frac{L - \varphi_2^k a_2^k}{\varphi_2^{k-1}}, \qquad h_k h_{k-1} = \frac{L - a_2^k \varphi_2^k}{\varphi_2^{k-1}}.$$

Thus if $h_k$ solves (37) then the one-form $\varphi = \sum \varphi_1^k \mathrm{d}a_1^k + \varphi_2^k \mathrm{d}a_2^k$ defined by

$$L - a_2^k \varphi_2^k = \varphi_1^k h_k, \qquad \varphi_1^{k+1} = \varphi_2^k h_k, \tag{45}$$

where $L$ is a constant, is in the kernel of the Poisson pencil.

2. The above one-form $\varphi$ is exact and its primitive is $L\ln(h_1 \cdots h_d)$. Indeed, by (37) and (45) we can write

$$\frac{L}{h_k} = \varphi_1^k + (h_{k+1} - (a_1^{k+1} + \lambda))\varphi_2^k.$$

Using this expression, we obtain, for any curve $a(t)$ in $\mathcal{N}$,

$$\begin{aligned}
\langle \varphi, \dot{a} \rangle &= \sum_{k=1}^{d} (\varphi_1^{k+1} \dot{a}_1^{k+1} + \varphi_2^k \dot{a}_2^k) = \sum_{k=1}^{d} (\varphi_2^k h_k \dot{a}_1^{k+1} + \varphi_2^k \dot{a}_2^k) \\
&= \sum_{k=1}^{d} \varphi_2^k (\dot{h}_k h_{k+1} + h_k \dot{h}_{k+1} - (a_1^{k+1} + \lambda)\dot{h}_k) \\
&= \sum_{k=1}^{d} \varphi_2^k \dot{h}_k h_{k+1} + \sum_{k=1}^{d} \varphi_2^k h_k \dot{h}_{k+1} - \sum_{k=1}^{d} \left( \varphi_1^k - \frac{L}{h_k} + \varphi_2^k h_{k+1} \right) \dot{h}_k \\
&= \sum_{k=1}^{d} \varphi_2^k \dot{h}_k h_{k+1} - \sum_{k=1}^{d} \varphi_2^k \dot{h}_k h_{k+1} + \sum_{k=1}^{d} \varphi_2^k h_k \dot{h}_{k+1} - \sum_{k=1}^{d} \left( \varphi_1^k - \frac{L}{h_k} \right) \dot{h}_k \\
&= \sum_{k=1}^{d} \varphi_1^{k+1} \dot{h}_{k+1} - \sum_{k=1}^{d} \left( \varphi_1^k - \frac{L}{h_k} \right) \dot{h}_k = \sum_{k=1}^{d} \frac{L}{h_k} \dot{h}_k = \frac{d}{dt}(L\ln(h_1 \cdots h_d)).
\end{aligned}$$

Therefore $L\ln(h_1 \cdots h_d)$ is a Casimir and so is the product $C = h_1 \cdots h_d$.

3. To express the solutions $h_k$ of the characteristic equation as Laurent series, it suffices to write $h_k$ in either of the following forms:

$$h_k = \lambda + \sum_{j=1}^{+\infty} \frac{h_{k,j}}{\lambda^j}, \qquad h_k = \sum_{j=-1}^{+\infty} \frac{h_{k,j}}{\lambda^j},$$

and substitute in (37). The coefficients $h_{k,j}$ are then determined algebraically.

□

We call (37) the *characteristic equation*. It is the counterpart of the well-known Riccati equation $h^2 + h_x = u + \lambda$ that appears in the study of the KdV equation (see, e.g., [5]). Proposition 4 allows us in principle to calculate the Casimirs of the Poisson pencil $\mathfrak{p}_\lambda$, but the computation is lengthy. Fortunately there is a shortcut: if in (37) we set

$$h_k = \frac{\psi_{k+1}}{\psi_k}\mu, \tag{46}$$

the characteristic equation becomes the linear system

$$\mu^2 \psi_{k+2} - (a_1^{k+1} + \lambda)\psi_{k+1}\mu - a_2^k \psi_k = 0. \tag{47}$$

We can express (47) in matrix form as

$$L\psi = 0, \tag{48}$$

where $L$ is the Lax matrix

$$L = \begin{pmatrix} \mu(a_1^1+\lambda) & -\mu^2 & 0 & & a_2^d \\ a_2^1 & \mu(a_1^2+\lambda) & -\mu^2 & \ddots & \\ 0 & a_2^2 & \ddots & \ddots & 0 \\ & \ddots & \ddots & \mu(a_1^{d-1}+\lambda) & -\mu^2 \\ -\mu^2 & 0 & & a_2^{d-1} & \mu(a_1^d+\lambda) \end{pmatrix},$$

and $\psi$ is the vector of the "homogeneous coordinates"

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_d \end{pmatrix}.$$

For (47) to admit non trivial solutions we must have $\det L = 0$. It can be proved that the cyclicity of the matrix $L$ implies that its determinant is a polynomial of degree 2 in $\mu^d$. Therefore we must have

$$0 = \det L = -\mu^{2d} + C_1\mu^d + C_2, \tag{49}$$

where $C_1$ is a monic polynomial of degree $d$ in $\lambda$

$$C_1(\lambda) = H_1 + \lambda H_2 + \cdots + \lambda^{d-2} H_{d-1} + \lambda^{d-1} K_1 + \lambda^d, \tag{50}$$

for proper coefficients $H_i$, $K_1$. The function $C_2$ instead does not depend on $\lambda$

$$C_2 = K_2 = (-1)^{d+1} a_2^1 a_2^2 \cdots a_2^d. \tag{51}$$

By Proposition 4 and Eq. (46), for all $\mu$ that satisfy (49) we have that $\mu^d = h_1 \cdots h_d$ is a Casimir, thus $C_1$ and $C_2$ are Casimirs as well. Their $(d+1)$ coefficients provide all information about the geometry of the system at hand. Indeed, since $K_1$ and $K_2$ are Casimirs of the Poisson tensor $\mathfrak{p}'$, the $2(d-1)$-dimensional manifold described by the equations

$$K_1 = \text{constant}, \qquad K_2 = \text{constant}$$

is a symplectic leaf of $\mathfrak{p}'$.[3] Furthermore, due to the Lenard relations, the $(d-1)$ coefficients $H_i$ are a set of functions in involution with respect to the Poisson bracket induced by $\mathfrak{p}'$. Since these functions are independent, the symplectic leaf admits a complete set of functions in involution and thus the vector fields

$$X_i = \mathfrak{p}' \mathrm{d} H_i$$

describe an integrable system. This is the $d$-particle periodic Toda hierarchy.

### *4.1. An example: the 3-particle Toda lattice*

To illustrate how the scheme described above works and how the Toda lattice arises we consider here the specific case where $d = 3$. This example is easy to handle, but at the same time it displays all the features of the general case. In order to make the equations easier to read we will change notation: we set $a_1^k = b_k$, $a_2^k = a_k$, $\varphi_1^k = \beta_k$, $\varphi_2^k = \alpha_k$. Thus our bi-Hamiltonian manifold $\mathcal{N}$ becomes $\mathbb{R}^6$ with coordinates $(a_1, a_2, a_3, b_1, b_2, b_3)$. The Poisson pencil $\mathfrak{p}_\lambda$ associates with a one-form $\sum_{k=1}^3 (\alpha_k da_k + \beta_k db_k)$ the vector field

$$\begin{aligned} \dot{a}_k &= a_k(-\alpha_{k+1}a_{k+1} + \alpha_{k-1}a_{k-1} - \beta_{k+1}(b_{k+1} + \lambda) + \beta_k(b_k + \lambda)), \\ \dot{b}_k &= (b_k + \lambda)(-\alpha_k a_k + \alpha_{k-1}a_{k-1}) + a_k\beta_{k+1} - a_{k-1}\beta_{k-1}, \end{aligned} \tag{52}$$

where $k = 1, 2, 3$. Eqs. (50) and (51) provide the Casimirs of the Poisson pencil (52)

$$C_1(\lambda) = H_1 + \lambda H_2 + \lambda^2 K_1 + \lambda^3, \qquad C_2(\lambda) = K_2, \tag{53}$$

where

$$\begin{aligned} &H_1 = b_1b_2b_3 + a_2b_1 + a_1b_3 + a_3b_2, \quad H_2 = b_1b_2 + b_1b_3 + b_2b_3 + a_2 + a_1 + a_3, \\ &K_1 = b_1 + b_2 + b_3, \qquad K_2 = a_1a_2a_3. \end{aligned} \tag{54}$$

Therefore the two vector fields $X_1 = \mathfrak{p}'\mathrm{d}H_1$ and $X_2 = \mathfrak{p}'\mathrm{d}H_2$ read

$$\begin{aligned} X_1: \quad \dot{a}_k &= a_k(b_{k+1}b_{k-1} - b_{k-1}b_k + a_{k+1} - a_{k-1}), \\ \dot{b}_k &= b_{k+1}a_{k-1} - b_{k-1}a_k, \end{aligned} \tag{55}$$

$$X_2: \quad \dot{a}_k = a_k(b_{k+1} - b_k), \qquad \dot{b}_k = a_{k-1} - a_k. \tag{56}$$

[3] One can show (see [18]) that such a symplectic leaf can be endowed with a bi-Hamiltonian structure, obtained by a Marsden–Ratiu reduction from the one on $\mathcal{N}$.

They define an integrable system on the symplectic leaves of $\mathfrak{p}'$, which are four-dimensional phase spaces. Expressions (55) and (56) are the equations of the 3-particle periodic Toda system.

## 5. Conclusions

In this paper we defined a new type of manifolds, which we called the bi-anchored Poisson manifold. We showed that a suitable reduction of a specific instance of these manifolds, namely the set of maps from $\mathbb{Z}_d$ to $GL(2,\mathbb{R})$, gives rise to the phase space of the $d$-particle periodic Toda lattice. This way we saw from a new point of view how the periodic Toda lattice can be considered the discrete counterpart of the KdV hierarchy.

There are several further developments of this approach. First of all, it is easy to endow the set of maps from $\mathbb{Z}_d$ to $GL(n,\mathbb{R})$ for a generic $n \in \mathbb{N}$ with the structure of "bi-anchored Poisson manifold". It is possible to show that the reduction of these manifolds gives rise to the other periodic Toda systems, which are the discrete analog of Gelfand–Dickey hierarchies [6]. This has been verified for the case $n = 3$ in [15].

Secondly (see [12,15]) the study of the conservation laws of the periodic Toda lattice allows to define the discrete analog [8] of the KP equations on Sato Grassmannian [19]. These represent flows on an infinite-dimensional phase space that admit invariant submanifolds. These submanifolds are the different phase spaces of the full periodic Toda lattice, and the restriction of the KP equation to these phase spaces are the Toda equations. This way it is possible to extend to the discrete case the description given for the continuous case in [5], where the KdV hierarchy and the (usual) KP equations are considered.

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